ON THE VIBRATION THEORY OF QUASILINEAR SYSTEMS WITH LAG

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This article generalizes the vibration theory of quasilinear systems whose motion is described by ordinary differential equations to systems with a time lag. The case of resonance in quasilinear non-autonomous systems with a lag is considered.

1. Formulation of the problem. We shall consider a system whose motion is described by differential-difference equations of the form

$$\frac{dx}{dt} = \sum_{\sigma=1}^{r} a_\sigma x(t-\tau_\sigma) + f(t) + \mu X(t,\ x(t-\tau_1), \ldots,\ x(t-\tau_r),\ \mu) \qquad (1.1)$$

Where

$$x(t) = (x_1(t), \ldots,\ x_n(t)), \qquad a_\sigma = \|a_{\sigma sj}\| \quad (s,\ j = 1, \ldots, n)$$

a_σ are constant matrices, $f(t) = f_1(t), \ldots, f_n(t))$ are periodic and continuous functions of time t, of period 2π, and the functions $X = (X_1, \ldots, X_n)$ are periodic and continuous with respect to t, of period 2π. These functions have continuous partial derivatives with respect to $x_1(t-\tau_1), \ldots, x_n(t-\tau_r)$ in some region G, defined by the inequalities $|x(t = \tau_\sigma)| < R$, $|\mu| < \mu^*$ where R and μ^* are positive constants. The functions X_s have, in the same region, continuous partial derivatives with respect to the parameter μ. The positive constants $\tau_1, \ldots, \tau_r$ are such that

$$\tau_1 = 0 < \tau_2 < \tau_3 < \cdots < \tau_r < 2\pi$$

The problem is to determine the periodic solutions (of period 2π) of the system (1.1) which become, when $\mu = 0$, the periodic solution $x_s^{(0)}$ of the generating system

$$\frac{dx(t)}{dt} = \sum_{\sigma=1}^{r} a\ x(t-\tau_\sigma) + f(t) \qquad (1\ 2)$$

We will consider the characteristic equation

$$\Delta(\lambda) = \left| \sum_{\sigma=1}^{r} a_\sigma e^{-\tau_\sigma \lambda} - E\lambda \right| = 0 \tag{1.3}$$

of the homogeneous system (1.2).

The roots of this equation which are either equal to zero or to $\pm N_j \sqrt{-1}$ will be called critical. (N_j are integers).

We will distinguish two cases: the non resonance case, when equation (1.3) has no critical roots or roots near (to a magnitude of order of smallness μ) to numbers of the form $\pm N_j \sqrt{-1}$, and the resonance case, when among the roots of equation (1.3) some of the roots are critical. In both cases the problem will be the determination of the periodic solutions of the system (1.1).

This is the way in which the problem was stated in articles [1-3] when quasi-linear systems whose motion is described by ordinary differential equations were considered. The results of these articles can, however, be carried over with great generality to systems with lag.

2. Periodic solutions of the system (1.2). *1. The non-resonance case.* (a) Let us assume that all roots of the equation (1.3) have real parts which are different from zero.

We will introduce into the discussion the functions

$$\Gamma_{js}(i\lambda) = \frac{\Delta_{js}(i\lambda)}{\Delta(i\lambda)} \tag{2.1}$$

where $\Lambda(i\lambda)$ is defined by formula (1.3), and $\Lambda_{js}(i\lambda)$ is the algebraic cofactor of the element of the matrix $\|\Lambda(i\lambda)\|$ in the jth column and sth row. It is clear that under the assumption made about the roots of equation (1.3) the functions $\Gamma_{js}(i\lambda)$ will be continuous in the interval $-\infty < \lambda < \infty$, and as $|\lambda| \to 0$, will have order $O(|\lambda|^{-1})$. Moreover, the functions $\Gamma'_{js}(i\lambda)$ will also be continuous and satisfy the condition $O(|\lambda|^{-1})$.

We will define the functions

$$K_{js}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\lambda}\Gamma_{js}(i\lambda)\,d\lambda, \qquad \overset{\circ}{K}_{js}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\lambda}\Gamma'_{js}(i\lambda)\,d\lambda \tag{2.2}$$

In accordance with the remarks made about the properties of the functions $\Gamma_{js}(i\lambda)$, it can be shown that the integrals

$$\int_{-\infty}^{+\infty} |K_{js}(t)|\,dt \tag{2.3}$$

have finite values.

The periodic solution of the system (1.2) is defined by the following functions:

$$x_s^*(t) = \sum_{j=1}^{n} \int_{-\infty}^{+\infty} f_j(x)\,K_{sj}(x-t)\,dx \equiv \sum_{j=1}^{n} \int_{-\infty}^{+\infty} f_j(t+x)\,K_{sj}(x)\,dx \tag{2.4}$$
$$(s = 1, \ldots, n)$$

This periodic solution will be unique. It can, of course, always be found in the form of a Fourier series, if we assume that the functions $f(t)$ have Fourier expansion. The estimate

$$|x_s^*(t)| < AM, \qquad A = \max\left\{\sum_{j=1}^{n} \int_{-\infty}^{+\infty} |K_{sj}(x)|\,dx\right\} \tag{2.5}$$

holds, where M is subject to the condition $|f_j(t)| < M$.

(b) We will assume now that we are dealing with the non-resonance case, but that among the roots of equation (1.3) there is a finite number of simple pure imaginary noncritical roots of the form $\omega_j i$, where $\omega_j \neq N_j$ (N_j are integers). In this case the system (1.1) will also have a unique periodic solution, conforming to an estimate of type (2.5).

To construct the solution, we will define the functions $\Gamma_{sj}^*(i\lambda)$ as follows: we will surround the numbers ω_j by the intervals 2ϵ which are so small that the intervals $[\omega_j - \epsilon, \omega_j + \epsilon]$ will not contain any integer. This can always be accomplished, since $\omega_j\sqrt{-1}$ are noncritical roots. Let outside these intervals and on their boundaries $\Gamma_{sj}^*(i) = \Gamma_{sj}(i\lambda)$. Inside the indicated intervals we define the functions Γ_{sj}^* such that the functions, together with their first derivatives with respect to λ, will be continuous in the interval $-\infty < \lambda < \infty$. This can always be done. Then the periodic solution $x_s^*(t)$ of the system (1.1), will also, as before, in this case determine the relations (2.2) and (2.4), in which the $\Gamma_{sj}(i\lambda)$ are replaced by the $\Gamma_{sj}^*(i\lambda)$. For this solution the estimates (2.5) are valid.

2. *The resonance case*. We will assume now that among the roots of equation (1.3) there is a finite number of simple critical roots of the form $N_j\sqrt{-1}$ $(j = 1, \ldots, k)$ and that the remaining roots are noncritical and satisfy either condition "a" or condition "b". In this case the periodic solution of the system (1.2) can be broken up into two parts; one part will have no critical harmonics $N_j\sqrt{-1}$, and the other solution will consist of critical harmonics only. Speaking generally, it is natural

that in this case a periodic solution may not exist. In order that the system (1.2) have periodic solutions, the functions $f_j(r)$ must satisfy conditions which are similar to the case of ordinary differential equations.

Let

$$f_j(t) = \varphi_j(t) + \sum_{l=1}^{k} c_{jl} \exp(N_l t \sqrt{-1}) \qquad (j = 1, \ldots, n) \tag{2.6}$$

Here the functions $\phi_j(t)$ have no resonance harmonics

$$c_{jl} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f_j(t) \exp(-N_l t \sqrt{-1})\, dt \qquad (j = 1, \ldots, n;\ l = 1, \ldots, k) \tag{2.7}$$

We will seek the periodic solution $x_s^*(t)$ of the system (1.2) in the form of the sum

$$x_s^* = x_{s1}^* + x_{s2}^*$$

where the x_{s1}^* are determined from the system (1.1), in which the $f_s(t)$ are replaced by the $\phi_s(t)$, and where the s_{s2}^* are determined from the system (1.1) in which the functions $f_s(t)$ are replaced by the second term of (2.6).

We will construct the functions $\Gamma_{sj}^*(t)$ as in "b", supplementing the intervals $(\omega_i - \epsilon,\ \omega_i + \epsilon)$ by the additional intervals $(N_j - \epsilon,\ N_j + \epsilon)$. Outside these intervals $\Gamma_{sj}^*(i\lambda) = \Gamma_{sj}(i\lambda)$. The $\Gamma_{sj}^*(i\lambda)$ are constructed in the interior of the indicated intervals so that the functions and their derivatives with respect to λ are continuous in the interval $-\infty < \lambda < +\infty$. Then the periodic functions x_{s1}^* will be determined using formulas (2.2) and (2.4), in which Γ_{sj} must be replaced by Γ_{sj}^*, and the functions $f_s(t)$ must be replaced by the periodic functions $\phi_s(t)$.

The functions $\phi_s(t)$ have the estimates

$$|\varphi_s(t)| < M(1 + k) = M^* \qquad (M > |f_s(t)|) \tag{2.8}$$

where M are positive constants. The estimates (2.5) will hold for the periodic functions s_{s1}^*, in which M and Γ_{sj} must be replaced by M^* and Γ_{sj}^* respectively. We will note that the estimate (2.8) can be substantially improved if we assume differentiability with respect to t of the functions $f(t)$.

The periodic functions x_{s2}^* will be sought in the form of a trigonometric series

$$x_{s2}^*(t) = \sum_{j=1}^{k} C_{sj} \exp(N_j \sqrt{-1}\, t) \qquad (s = 1, \ldots, n) \tag{2.9}$$

The constants C_{sj} satisfy a system of linear homogeneous algebraic equations of the form

$$\sum_{l=1}^{n}\Big(\sum_{\sigma=1}^{r} a_{\sigma sl}\exp(-\tau_\sigma N_j\sqrt{-1})-\delta_{sl}N_j\sqrt{-1}\Big)C_{lj}=c_{sj}\qquad\begin{pmatrix}s=1,\ldots,n\\ j=1,\ldots,k\end{pmatrix}\tag{2.10}$$

The system (2.10) has a solution, and the system (1.2) admits a periodic solution, provided the conditions

$$\int_0^{2\pi}\sum_{l=1}^{n} b_{lj}\exp(-N_j\sqrt{-1})f_l(t)\,dt=0\qquad (j=1,\ldots,k)\tag{2.11}$$

are satisfied.

Here b_{lj} exp $(-N_j\sqrt{-1})$ $(j=1,\ldots,k)$ are period solutions of the "conjugate" system of the homogeneous system (1.2), of the form

$$\frac{dx}{dt}=-\sum_{\sigma=1}^{r} a_\sigma' x(t+\tau_\sigma)\tag{2.12}$$

where a_σ' is the transposition of the matrix a_σ. It is clear that b_{lj} satisfies the following system of linear homogeneous equations

$$\sum_{l=1}^{n}\Big(\sum_{\sigma=1}^{r}-a_{\sigma ls}\exp(\tau_\sigma N_j\sqrt{-1})-\delta_{sl}N_j\sqrt{-1}\Big)b_{lj}=0\qquad (s=1,\ldots,n)\tag{2.13}$$

We will assume that the conditions (2.11) are satisfied. Then the system (2.10) is compatible. The solution can be found as follows.

Since $N_j\sqrt{-1}$ is a simple root of the equation (1.3), we can find among the first minors at least one minor which is different from zero. We will assume that this minor corresponds to the element in the intersection of the first row with the first column $\Delta_{11}\neq 0$. Ignoring the first equation and setting $C_{1j}=0$, we shall find the remaining $C_{2j},\ldots,C_{nj}$ by Cramer's rule from the last $n-1$ of equation (2.10). It is easily seen that all C_{sj} will have the estimate

$$|C_{sj}|<B_jM\qquad (M>|f_j(t)|)$$

where B_j depends on the form of the matrix a_σ and the root $N_j\sqrt{-1}$ of the equation (1.3).

Therefore the solution x_{s2}^* will be completely determined by the determined operator L_s^*, which depends on f:

$$x_{s2}^*=L_s^*(t,\ f)\tag{2.14}$$

This operator has the properties

$$L_s^*(t,f^{(1)}+f^{(2)})=L_s^*(t,f^{(1)})+L_s^*(t,f^{(2)})\tag{2.15}$$

$$L_s^*(t, cf) = cL_s^*(t, f) \qquad (c = \text{const}) \tag{2.16}$$

$$|L_s^*(t, f)| < \sum_{j=1}^{k} B_j M \tag{2.17}$$

Finally, the linear homogeneous system (1.2), in the resonance case, has a periodic solution of period 2π of the form

$$x_s = \sum_{j=1}^{k} M_j d_{sj} \exp(N_j \sqrt{-1} t) \qquad (s = 1, \ldots, n) \tag{2.18}$$

where $M_1, \ldots, M_k$ are arbitrary constants d_{sj} – k of particular solutions of the k linear homogeneous system (2.10) (in which $c_{sj} = 0$).

The particular solution $x_{s1}^*(t)$, which has been found earlier, can also be represented by means of a completely determined operator L_s^{**}, satisfying the conditions (2.15), (2.16) and the estimate

$$L_s^{**}(t, \varphi)| < AM \tag{2.19}$$

where the constants are determined in the formulas (2.5), (2.8).

Introducing the determined operator $L_s = L_s^* + L_s^{**}$, we come to the following conclusion.

In the resonance case, when the equation (1.3) has k simple critical roots, a periodic solution of the system (1.2) exists, provided the conditions (2.11) are satisfied.

This periodic solution will have k arbitrary constants $M_1, \ldots, M_k$, and can be written in the form

$$x_s^*(t) = \sum_{j=1}^{k} M_j \varphi_{sj}(t) + L_s(t, f_1, \ldots, f_n), \qquad (s = 1, \ldots, n) \tag{2.20}$$

where ϕ_{sj} are periodic solutions of the homogeneous system (1.2), and L_s is a completely determined operator satisfying the conditions (2.15), (2.16) and the estimate

$$|L_s(t, f_1, \ldots, f_n)| < \left(\sum_{j=1}^{k} B_j + A(1+k)\right) M = A_1 M \tag{2.21}$$

Note. In the case when the critical roots $N_j \sqrt{-1}$ are multiple, the number of periodic solutions of the homogeneous systems (1.2) and (2.12) will be $m < k$. The conditions (2.11) will, as before, be existence conditions for a solution of the system (1.2) but there will be m of them. As these conditions are satisfied, the periodic solution can, as before, be written in the form (2.20). Here, however, the number of arbitrary constants M_i will be m, and the operators L_s will have another form, but they will still have the same properties of linearity and homogeneity and an estimate of type (2.21).

3. The periodic solution of the system (1.1) in the non-resonance case. *Theorem 1.* If among the roots of the characteristic equation there are no critical roots, and if the number of pure imaginary non-critical roots is finite, then the system (1.1) has a unique periodic solution of period 2π, defined in the region $|\mu| \leqslant \eta$ (η is a sufficiently small positive number) and becomes, when $\mu = 0$, the generating periodic solution of the system (1.2).

The proof of Theorem 1 is easily obtained by the usual method of successive approximations.

We will note that the theorem is also valid for an infinite number of imaginary non-critical roots, provided we can find an $\epsilon > 0$, such that in the intervals $(\omega_j - \epsilon,\ \omega_j + \epsilon)$ there will be no imaginary non-critical roots on the imaginary axis.

4. The periodic solution of the system (1.1) in the resonance case. We will assume that the system (1.2) admits a generating periodic solution of the form

$$x_s^0 = M_1^0\varphi_{s1}(t) + \ldots + M_k^0\varphi_{sk}(t) + \varphi_s(t) \qquad (s = 1, \ldots, n) \tag{4.1}$$

where $M_1^0, \ldots, M_k^0$ are constants, ϕ_{si} are particular periodic solutions of the homogeneous system (1.2), and $\phi_s(t)$ is the periodic solution of the system (1.2). The latter holds if the conditions (2.11) are satisfied. We will assume that the constants $M_j^{(0)}$ are such that $\{x_s^{(0)}\}$ lies in the region G. Then the following theorem, which generalizes the propositions of Malkin [2] to systems with lag, is valid.

Theorem 2. A necessary condition that the system (1.1) have a periodic solution $x_s(t, \mu)$ which becomes, when $\mu = 0$, the generating solution of (4.1), is the requirement that the constants $M_1^{(0)}, \ldots, M^{(0)}$ satisfy the system of equations

$$P_j(M_1^0, \ldots, M_k^0) \equiv -\frac{1}{2\pi}\int_0^{2\pi}\sum_{s=1}^{n} X_s(t,\ x^{(0)}(t-\tau_1), \ldots, x^{(0)}(t-\tau_r), 0)\,\psi_{sj}\,dt$$

$$(j = 1, \ldots, k) \tag{4.2}$$

where the ψ_{sj} are periodic solutions of the "conjugate" of the system (2.13).

If under these conditions the Jacobian

$$\frac{\partial(P_1, \ldots, P_k)}{\partial(M_1, \ldots, M_k)} \neq 0 \quad \text{for } M = M^{(0)} \tag{4.3}$$

is different from zero, then the system (1.1) has a unique periodic solution $x_s(t, \mu)$, which becomes, when $\mu = 0$, the generating periodic solution of (4.1).

The solution is defined in the region $|\mu| \leqslant \eta$ where $\eta > 0$ is sufficiently small. This proposition is a simple consequence of the more general condition for the existence of a periodic solution for the system (1.1), and it generalizes the conditions which we obtained in article [3]. We will derive these conditions.

5. The auxiliary system and its periodic solutions. 1. We will consider the following system of equations:

$$\frac{dx}{dt} = \sum_{\sigma=1}^{r} a_\sigma x(t-\tau_\sigma) + f(t) + \sum_{j=1}^{k} \varphi_j(t) W_j \tag{5.1}$$

where $\phi_j = \{\phi_{1j}, \ldots, \phi_{nj}\}$ are periodic solutions of the homogeneous system and the W_j are constants.

We will show that the constants W_j can always be chosen so that the conditions for the existence of a periodic solution of the differential system (5.1) will always be satisfied.

In fact, for the system (5.1), these conditions have the form

$$\int_0^{2\pi} \sum_{s=1}^{n} \psi_{sj} f_s(t)\, dt + d_{1j} W_1 + \ldots + d_{kj} W_k = 0 \qquad (j=1,\ldots,k) \tag{5.2}$$

where ψ_{sj} are periodic solutions of the "conjugate" of the system (2.12), and d_{ij} is defined by the equations

$$d_{ij} = \int_0^{2\pi} \sum_{s=1}^{n} \varphi_{si} \psi_{sj}\, dt \tag{5.3}$$

It is sufficient to show that the determinant $|d_{ij}|$ $(i, j = 1, \ldots, n)$ is different from zero.

If $d_{i1} = \ldots = d_{ik} = 0$, then this would mean that to the critical root $N_i \sqrt{-1}$ there correspond not one, but two particular solutions. Therefore, not all the d_{ij} are equal to zero.

We will assume now that $|d_{ij}| = 0$. We will show that this leads to a contradiction. In fact we can find numbers $\Lambda_1, \ldots, \Lambda_k$ such that

$$\Lambda_1 d_{1j} + \ldots + \Lambda_k d_{kj} = 0 \qquad (j = 1, \ldots, k) \tag{5.4}$$

But then the system of equations

$$\frac{dx}{dt} = \sum_{\sigma=1}^{r} a_\sigma x(t-\tau_\sigma) + \Lambda_1\varphi_1 + \ldots + \Lambda_k\varphi_k \tag{5.5}$$

will have a periodic solution $\phi(t)$, and the homogeneous system (1.2) will have, in addition to the k periodic solutions corresponding to the k critical roots, a solution with secular terms which also corresponds to the critical roots:

$$x(t) = \varphi(t) + (\Lambda_1\varphi_1(t) + \ldots + \Lambda_k\varphi_k(t))\,t$$

The latter is impossible by virtue of the assumptions that we have made, since $x(t)$ is independent of $\phi_1, \ldots, \phi_k$, and is the $k+1$ solution corresponding to k critical roots. Therefore the determinant $|d_{ij}|$ is different from zero.

2. We will consider the auxiliary system of integro-differential equations with lag having the form

$$\frac{dx(t)}{dt} = \sum_{\sigma=1}^{r} a_\sigma x(t-\tau_\sigma) + f(t) + \mu X(t, x(t-\tau_1), \ldots x(t-\tau_r), \mu) + \sum_{j=1}^{k} \varphi_j W_j \tag{5.6}$$

Here the constants W_j are uniquely determined from the linear non-homogeneous system

$$\mu \int_0^{2\pi} X(t, x(t-\tau_1), \ldots, x(t-\tau_r), \mu)\phi_j dt + \sum_{i=1}^{k} W_i d_{ij} = 0 \qquad (j = 1, \ldots, k) \tag{5.7}$$

We assume that the conditions (2.11) are satisfied for f_s.

Lemma. The system of integro-differential equations with lag determines a family of periodic solutions depending on k arbitrary constant parameters $M_1, \ldots, M_k$ and a parameter μ of the form

$$x_s^*(t, M, \mu) = M_1\varphi_{s1} + \ldots + M_k\varphi_{sk} + \varphi_s + \mu x_s^*(t, M_1, \ldots, M_k, \mu) \qquad (s = 1, \ldots, k) \tag{5.8}$$

where $x_s^*(t, M, \mu)$ are continuous functions of the parameters $M_1, \ldots, M_k$ defined in some neighborhood of the fixed point $M_1^{(0)}, \ldots, M_k^{(0)}$, and the parameter μ, for $|\mu| \leqslant \mu^*$ (μ^* is some positive number). These functions have continuous partial derivatives with respect to $M_1, \ldots, M_k$. They are periodic and continuous functions of time t and period 2π.

When X_s are analytic relative to x and μ in G, the functions x^* will also be analytic relative to $M_1, \ldots, M_m$ in some neighborhood of the point $M_1^{(0)}, \ldots, M_m^{(0)}$ for $|\mu| \leqslant \mu^*$.

The proof of this proposition is obtained by the method of successive approximations.

We will take for the first approximation the generating periodic solution $x_s^{(0)}$ and the constants $W_j^{(0)}$ determined from the system (5.6) and (5.7) when $\mu = 0$. We find that the $x_s^{(0)}$ are determined using formulas

(4.1) and $W_j^{(0)} = 0$. Then the lth approximation is determined from the system of linear equations

$$\frac{dx^{(l)}}{dt} = \sum_{\sigma=1}^{r} a_\sigma x^{(l)}(t-\tau_\sigma) + f(t) + \mu X(t, x^{(l-1)}(t-\tau_1),\ldots, x^{(l-1)}(t-\tau_r), \mu) + \\ + \sum_{j=1}^{k} \varphi_j W_j^{(l)}$$

$$\mu\int_0^{2\pi} X(t, x^{(l-1)}(t-\tau_1),\ldots, x^{(l-1)}(t-\tau_r), \mu)\psi_j\,dt + \sum_{i=1}^{k} W_i^{(l)} d_{ij} = 0$$

In proving that all approximations lie in the region G, as well as in the construction of majoring series, we will use the estimates obtained in Section 2 of this article.

The proof of the convergence of the sequences $x^{(l)}$ and $W_i^{(l)}$ yields an estimate for the number μ^*.

6. Necessary and sufficient conditions for the existence of periodic solutions of the system (1.1). We will assume that the periodic solution x_s^* of the auxiliary system (5.6), (5.7) has been found. The corresponding constants will be found from the equations (5.7).

Let us introduce the notation

$$W_j^* \equiv \mu P_j^* \;(M_1,\ldots,M_k,\mu) = -\frac{\mu}{2\pi}\int_0^{2\pi}\sum_{s=1}^{n} X_s(t, x^*(t-\tau_1)\ldots x^*(t-\tau_r), \mu)\psi_{sj}\,dt \qquad (j=1,\ldots,k) \tag{6.1}$$

The functions P_i^* as well as x_s^* and W_j^* will be defined in some region

$$|\mu| \leqslant \mu^*, \quad |M_i - M_i^{(0)}| \leqslant H \quad (i=1,\ldots,k)$$

where H is some positive number determined in the course of the proof of the convergence of the successive approximations $x^{(l)}$, $W^{(l)}$, so that the $x^{(l)}$ lie in the region G.

Theorem 3. In order that the system (1.1) have a periodic solution of period 2π which becomes the generating solution, it is necessary and sufficient that the equations

$$P_j^*(M_1,\ldots, M_k, \mu) = 0 \qquad (j=1,\ldots,k) \tag{6.2}$$

have a solution $M_i(\mu)$ in some neighborhood $\mu \leqslant \eta_1 \leqslant \mu^*$ satisfying the condition $M_i(0) = M_i^{(0)}$.

Proof. Let the system (6.2) have a solution $M_j(\mu)$ $(M_j(0) = M_j^{(0)}$. Then the system of functions

$$x_s(t) = x_s^*(t, M_1(\mu), \ldots, M_k(\mu), \mu) \qquad (s = 1, \ldots, n)$$

will be the periodic solution of the system (1.1). This proves the sufficiency.

We will assume that the system (1.1) has a periodic solution of the form indicated. Then the solution must belong to the family (5.8). Substituting in (5.6) and (5.7), we will find that the P_j^* must be equal to zero. This proves the necessity of the condition (6.2).

In particular, to insure that the system (1.1) have a periodic solution, it is necessary that the constants satisfy the equations (4.2). The necessity of the condition (4.3) follows from an application of implicit function theory to the equations (6.2).

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